Indian Statistical Institute
Mid-Semestral Examination
Differential Topology-MMath II

Max Marks: 40 Time: 3 hours

(1) Decide whether the following statements are true or false. Justify.

(a) The interval $[-1, 1]$ is diffeomorphic to

$$X = \{(x, y) \mid (0 \leq x \leq 1 \text{ and } y = 0) \text{ or } (x = 1 \text{ and } 0 \leq y \leq 1)\}.$$

(b) Every local diffeomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a diffeomorphism (onto its image).

(c) Every vector field on $\mathbb{R}$ is complete.

(d) The determinant map $M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is a Morse function for $n \geq 2$.

(e) Every smooth map $f : S^2 \longrightarrow S^1$ has a critical point. [$3 \times 5 = 15$]

(2) (a) Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. Find all $a \in \mathbb{R}$ so that

$$Z_a = \{p \in \mathbb{R}^n \mid p^t Ap = a\}$$

is a manifold. Find the dimension of Z_a when it is a manifold. [4]

(b) Let σ be an integral curve of a vector field X on a smooth manifold M. Suppose that $\dot{\sigma}(t) = 0$ for some t. Show that σ is a constant map. [4]

(c) Let $f : M \longrightarrow N$ be a smooth map. Smooth vector fields X on M and Y on N are said to be f-related if $df \circ X = Y \circ f$. Suppose that X, X_1 are smooth vector fields on M and Y, Y_1 smooth vector fields on N. (i) If X is f-related to Y and X_1 is f-related to Y_1 show that $[X, X_1]$ is f-related to $[Y, Y_1]$. (ii) Suppose that $df(X(p)) = df(X(q))$ whenever $f(p) = f(q)$. Is there a smooth vector field Z on N that is f-related to X? [4+3=7]

(3) (a) Define the term : *Morse function.* Let $f : U \longrightarrow \mathbb{R}$ be a smooth function defined on an open subset $U \subseteq \mathbb{R}^k$. For $x \in U$, let $H(x)$ denote the Hessian of f at x. Show that f is Morse if and only if

$$\det(H)^2 + \Sigma_i(\partial f/\partial r_i)^2 > 0$$

on U. [3]

(b) State the stability theorem. Show by examples that the properties (i) being an immersion, (ii) being transverse to a given submanifold are not stable properties of maps defined on non compact domains. [1+2+2=5]

(c) Decide whether there exists a bijective submersion $f : S^3 \longrightarrow \mathbb{RP}^2$. [2]